Modeling The Business Cycle
Part IV - Value Of The Debt Tax Shield

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We will define the debt tax shield value to be the present value of tax savings applicable to the tax-deductibility of interest payments on debt. In this white paper we will build a model that calculates the value of the debt tax shield for a company whose revenues are correlated with the business cycle. To that end we will work through the following hypothetical problem from Part III...

Our Hypothetical Problem

In Part III we were tasked with calculating the enterprise value of ABC Company. Enterprise value is defined as the present value of enterprise cash flow, which excludes the interest paid on debt financing. The table below presents ABC Company's go-forward model assumptions...

Table 1: Enterprise Value Assumptions From Part III

Description	Balance	Notes
Annualized revenue at time zero (in thousands)	\$10,000	Current revenue annualized
Annualized revenue growth rate (%)	5.00	Discrete-time secular growth rate (RGR)
Annualized revenue volatility (%)	25.00	Secular growth rate standard deviation
Assets as a percent of annualized revenue (%)	60.00	Total assets divided by annualized revenue
Return on assets (%)	13.50	After-tax ROA
Cost of capital (%)	12.00	Discrete-time annualized discount rate
Peak-to-trough change in revenue (%)	50.00	Excludes secular growth rate
Business cycle length in months	60	Peak-to-peak or trough-to-trough

The following table presents the go-forward assumptions applicable to ABC Company's interest-bearing debt financing and the value of the debt tax shield...

Table 2: Debt Financing Assumptions

Description	Balance
Ratio of debt to assets (%)	40.00
Interest rate on debt (%)	6.00
Income tax rate (%)	20.00

We are tasked with answering the following questions:

Question 1: What is the debt tax shield value at time zero given that cash flow is received over the infinite time interval $[0, \infty]$?

Question 2: What is debt tax shield value at the end of year 3 given that cash flow is received over the finite time interval $[3, 20]$?

Base Equations

The table below presents the enterprise value model parameters from Part III and the go-forward debt financing model parameters from Table 2 above...

Table 3: Model Parameter Values

Symbol	Description	Value
R_0	Current annualized revenue at time zero	\$10,000,000
λ	Continuous-time secular revenue growth rate	0.0488
κ	Continuous-time cost of capital	0.1133
ϵ	Ratio of total assets to annualized revenue	0.6000
β	Business cycle sine wave radians	1.2566
Δ	Business cycle sine wave amplitude	0.2500
ϕ	Current position in the business cycle (in years)	1.2500
ψ	Ratio of debt to assets	0.4000
ι	Debt interest rate	0.0600
τ	Income tax rate	0.2000

In Part III we defined the variable A_t to be total assets at time t. Using the parameters in Table 3 above the equation for expected total assets at time t from the perspective of time zero is... [3]

$$\mathbb{E}\Big[A_t\Big] = \epsilon\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \text{Exp}\Big\{\lambda\, t\Big\} \Big(1 + \Delta\, \sin(\beta\,(t + \phi))\Big) \tag{1}$$

Debt Tax Shield Value

We will define the variable $E_{a,b}$ to be expected pre-tax interest expense incurred over the time interval $[a, b]$, the variable ψ to be the ratio of debt to total assets, and the variable ι to be the debt interest rate. Using Equation (1) above the equation for pre-tax cumulative interest expense is...

$$E_{a,b} = \int_a^b \psi\, \mathbb{E}\Big[A_t\Big]\, \iota\, \delta t = \psi\, \iota \int_a^b \mathbb{E}\Big[A_t\Big]\, \delta t = \epsilon\, \psi\, \iota\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \int_a^b \text{Exp}\Big\{\lambda\, t\Big\} \Big(1 + \Delta\, \sin(\beta\,(t + \phi))\Big)\, \delta t \tag{2}$$

Note that we can rewrite Equation (2) above as...

$$E_{a,b} = \epsilon\, \psi\, \iota\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \Big(\int_a^b \text{Exp}\Big\{\lambda\, t\Big\}\, \delta t + \Delta \int_a^b \text{Exp}\Big\{\lambda\, t\Big\} \sin(\beta\,(t + \phi))\, \delta t\Big) \tag{3}$$

Using Appendix Equations (13) and (14) below we can rewrite Equation (3) above as...

$$E_{a,b} = \epsilon\, \psi\, \iota\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \Big(I(a, b)_1 + \Delta\, I(a, b)_2\Big) \text{ ...where... } \alpha = \lambda \tag{4}$$

Using Equations (2), (3) and (4) above the equation for the tax benefit applicable to the tax deductibility of interest expense is...

$$\text{Tax benefit} = \tau\, E_{a,b} \tag{5}$$

We will define the variable $V_{a,b}$ to be the present value at time a of the expected tax benefit to be realized over the time interval $[a, b]$

$$\begin{aligned} V_{a,b} &= \tau\, \psi\, \iota \int_a^b \mathbb{E}\Big[A_t\Big]\, \text{Exp}\Big\{-\kappa\,(t - a)\Big\}\, \delta t \\ &= \tau\, \epsilon\, \psi\, \iota\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \int_a^b \text{Exp}\Big\{\lambda\, t\Big\} \Big(1 + \Delta\, \sin(\beta\,(t + \phi))\Big) \text{Exp}\Big\{-\kappa\,(t - a)\Big\}\, \delta t \\ &= \tau\, \epsilon\, \psi\, \iota\, R_0 \Big(1 + \Delta\, \sin(\beta\, \phi)\Big)^{-1} \text{Exp}\Big\{\kappa\, a\Big\} \Big(\int_a^b \text{Exp}\Big\{(\lambda - \kappa)\, t\Big\}\, \delta t + \Delta \int_a^b \text{Exp}\Big\{(\lambda - \kappa)\, t\Big\} \sin(\beta\,(t + \phi))\, \delta t\Big) \end{aligned} \tag{6}$$

Using Appendix Equations (13) and (14) below we can rewrite Equation (6) above as...

$$V_{a,b} = \tau\,\epsilon\,\psi\,\iota\,R_0 \Big(1 + \Delta\,\sin(\beta\,\phi)\Big)^{-1} \,\mathrm{Exp}\Big\{\kappa\,a\Big\} \Big(I(a,b)_1 + \Delta\,I(a,b)_2\Big) \;\text{...where...}\; \alpha = \lambda - \kappa \tag{7}$$

The Answers To Our Hypothetical Problem

Question 1: What is the debt tax shield value at time zero given that cash flow is received over the infinite time interval $[0,\infty]$?

Using the data in Table 3 above and the Appendix Equations below the values of the following integrals are...

$$I(0,\infty)_1 = 15.4946 \;\text{...and...}\; I(0,\infty)_2 = 0.0408 \;\text{...where...}\; \alpha = \lambda - \kappa = -0.0645 \tag{8}$$

Using Equations (7) and (8) above and the data in Table 3 above the answer to the question is...

$$V_{0,\infty} = 0.20 \times 0.60 \times 0.40 \times 0.06 \times 10{,}000{,}000 \times \Big(1 + 0.25 \times \sin(1.2566 \times 1.25)\Big)^{-1}$$
$$\times\,\mathrm{Exp}\Big\{0.1133 \times 0\Big\}\Big(15.4946 + 0.25 \times 0.0408\Big) = 357{,}000 \tag{9}$$

Question 2: What is debt tax shield value at the end of year 3 given that cash flow is received over the finite time interval $[3,20]$?

Using the data in Table 3 above and the Appendix Equations below the values of the following integrals are...

$$I(3,20)_1 = 8.5052 \;\text{...and...}\; I(3,20)_2 = 0.3460 \;\text{...where...}\; \alpha = \lambda - \kappa = -0.0645 \tag{10}$$

Using Equations (7) and (10) above and the data in Table 3 above the answer to the question is...

$$V_{3,20} = 0.20 \times 0.60 \times 0.40 \times 0.06 \times 10{,}000{,}000 \times \Big(1 + 0.25 \times \sin(1.2566 \times 1.25)\Big)^{-1}$$
$$\times\,\mathrm{Exp}\Big\{0.1133 \times 3\Big\}\Big(8.5052 + 0.25 \times 0.3460\Big) = 278{,}000 \tag{11}$$

tau * epsilon * psi * iota * baseRevenue * Exp(kappa * a) * (mIOne + delta * mITwo)

Appendix

A. We will define the following equations... [4]

$$E_1 = \mathrm{Exp}\Big\{\alpha\,t\Big\} \;\text{...and...}\; E_2 = \mathrm{Exp}\Big\{\alpha\,t\Big\}\sin(\beta\,(t+\phi)) \tag{12}$$

B. Using the first equation in Equation (12) above we will make the following integral definition... [4]

$$I(a,b)_1 = \int_a^b E_1\,\delta t = \mathrm{Exp}\Big\{\alpha\,t\Big\}\,\alpha^{-1}\Big[_a^b \tag{13}$$

C. Using the second equation in Equation (12) above we will make the following integral definition... [4]

$$I(a,b)_2 = \int_a^b E_2\,\delta t = \mathrm{Exp}\Big\{\alpha\,t\Big\}\Big(\alpha\,\sin(\beta\,(t+\phi)) - \beta\,\cos(\beta\,(t+\phi))\Big)\Big(\alpha^2+\beta^2\Big)^{-1}\Big[_a^b \tag{14}$$

References

[1] Gary Schurman, *Modeling The Business Cycle - Part I,* October, 2020.

[2] Gary Schurman, *Modeling The Business Cycle - Part II,* October, 2020.

[3] Gary Schurman, *Modeling The Business Cycle - Part III,* October, 2020.

[4] Gary Schurman, *Modeling The Business Cycle - Mathematical Supplement,* October, 2020.